

An improved ε expansion for three-dimensional turbulence: summation of nearest dimensional singularities

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An improved ε expansion in the d -dimensional ($d > 2$) stochastic theory of turbulence is constructed by taking into account pole singularities at $d \rightarrow 2$ in coefficients of the ε expansion of universal quantities. Effectiveness of the method is illustrated by a two-loop calculation of the Kolmogorov constant in three dimensions.

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The renormalization-group (RG) method in the theory of turbulence is based on the stochastic Navier-Stokes equation with a Gaussian random force [1, 2, 3]. One of the central problems in this approach is calculation of the Kolmogorov constant C , i.e. the dimensionless amplitude in the scaling law [4]

$$S_2(r) = C(\bar{\varepsilon}r)^{2/3} \quad (1)$$

expressing the dependence of the second-order structure function $S_2(r)$ on the relative distance r in the inertial range $r_d \ll r \ll L$. Here, L is the external length of turbulence, r_d the dissipative length and $\bar{\varepsilon}$ the energy injection rate per unit mass (which, in the steady state, coincides with the dissipation rate).

Several attempts have been made in the past to solve this problem [5] - [14], but they all suffer from ambiguities in connecting model parameters and observable quantities. As a consequence, there are significant discrepancies in the predicted numerical values for the Kolmogorov constant (the spread is about a factor of two). In this Letter we analyse reasons of this unsatisfactory situation and present results of a calculation based both on an expression of C in terms of universal quantities and account of additional singularities arising near and below two dimensions. Rather unexpectedly, the analysis reveals that these singularities have a major effect on the numerical values of observable quantities well above two dimensions. We also show that a partial summation of these singularities is possible and significantly improves the numerical value obtained for C . To assess properties of the expansion produced within the RG approach, we have carried out the calculation in the two-loop approximation (the results of Refs. [5] - [14] were obtained in the one-loop approximation).

In the RG approach to d -dimensional turbulence a powerlike correlation function of the random force is often used: $\langle f f \rangle \sim D_0 k^{4-d-2\varepsilon} \equiv d_f(k)$. In the RG framework various quantities may be calculated in the form of an ε expansion which subsequently must be extrapolated to the physical value $\varepsilon = 2$. For some important

quantities the ε expansion breaks off, which for the function $S_2(r)$ yields the Kolmogorov exponent $2/3$ [as in Eq. (1)] at $\varepsilon = 2$. To find the Kolmogorov constant the amplitude of this function has to be calculated, which, however, can be done only approximately, because its ε expansion does not break off. In calculation of the amplitude, apart from technical difficulties at two-loop order, a principal problem arises as well: the answer for $S_2(r)$ has to be expressed in terms of the energy injection rate $\bar{\varepsilon}$ [as in Eq. (1)] instead of the parameter D_0 of the powerlike forcing function. This problem has been treated in different ways in Refs. [5] - [14] which has led to different one-loop values of C .

In Ref. [7] (see also [5, 6]) the connection between D_0 and $\bar{\varepsilon}$ was sought with the aid of the exact relation

$$\bar{\varepsilon} = \frac{(d-1)}{2(2\pi)^d} \int d\mathbf{k} d_f(k). \quad (2)$$

In the unphysical region $\varepsilon < 2$ this integral has to be cut off at wave numbers of the order of $\Lambda \equiv r_d^{-1}$. At fixed $\bar{\varepsilon}$ this procedure introduces, first, dependence on ε of the form $D_0 \sim (2-\varepsilon)$ in D_0 (which has to be taken into account in the construction of the ε expansion), and second, an ambiguity connected with the possibility to replace the upper limit Λ by $c\Lambda$ with an arbitrary coefficient c . The first feature is rather natural, because the powerlike forcing $d_f(k) \sim (2-\varepsilon) k^{4-d-2\varepsilon}$ reproduces in the limit $\varepsilon \rightarrow 2$ the realistic forcing by infinitely large eddies: $d_f(k) \sim \delta(\mathbf{k})$. The second feature, however, introduces arbitrariness in the sought connection between D_0 and $\bar{\varepsilon}$ through the coefficient $c^{2\varepsilon-4}$, which in turn renders the ε expansion of D_0 -dependent quantities ambiguous (in Ref. [7] the simplest choice $c = 1$ was used). This is a reflection of the fact that the physical content of the theory remains unaltered when D_0 is multiplied by an arbitrary function $F(\varepsilon)$ with $F(2) = 1$.

Another way to fix the connection between D_0 and $\bar{\varepsilon}$ has been used in Refs. [8] - [14]. It amounts to the use of an exact relation (for the physical value $\varepsilon = 2$ of the falloff exponent) which allows to connect the spectral energy flux with an integral of a third-order correlation

function, the latter being subsequently calculated in the form of an ε expansion. The use of this relation in the unphysical region $\varepsilon < 2$ is tantamount to a certain choice of the function $F(\varepsilon)$ mentioned above.

Thus, the ε expansion of the Kolmogorov constant in the model with the powerlike forcing is not unambiguous. Therefore, a better or worse agreement with the experimental value of C at one-loop level does not bear much meaning until a procedure for subsequent approximations has been pointed out and the stability of obtained results checked. On the other hand, since the real value of the expansion parameter $\varepsilon = 2$ is not small, it is difficult to expect good quantitative results without estimating – at least approximately – higher orders of the ε expansion.

In the model at hand, only quantities independent of D_0 have rigorously unambiguous dependence on ε (we will call them universal). Such quantities are, e.g., critical exponents and dimensionless ratios of structure functions $S_n(r)$, the skewness factor $\mathcal{S} \equiv S_3/S_2^{3/2}$ in particular. Calculation of universal quantities with the use of the RG method and the ε expansion yields unambiguous results and cannot lead to such "paradoxes" as different one-loop values for the Kolmogorov constant.

In view of this we have pursued the goal of finding a suitable universal quantity the physical value of which would be simply connected with the Kolmogorov constant, and calculating this quantity with the aid of the RG. The skewness factor \mathcal{S} , connected with the Kolmogorov constant through an exact relation $C = (-4/5\mathcal{S})^{2/3}$ [4, 15], might serve as such a quantity. However, in the unphysical region $\varepsilon < 2$ the structure function $S_2(r)$ in the model with the powerlike forcing correlation $d_f(k) \sim k^{4-d-2\varepsilon}$ contains – at $\varepsilon < 3/2$ – an independent of r UV-divergent additive term $\sim \Lambda^{2-4\varepsilon/3}$ [for $S_3(r)$ this problem is absent, see below]. As a consequence, a straightforward generalization of the skewness factor $\mathcal{S} \equiv S_3/S_2^{3/2}$ to the region $\varepsilon < 3/2$ becomes pointless, because the powers of r in this definition do not cancel due to the constant term in $S_2(r)$. Therefore, as the desired universal quantity we chose the "nearest relative" of the skewness factor, the quantity

$$Q(\varepsilon) \equiv \frac{r\partial S_2(r)/\partial r}{|S_3(r)|^{2/3}} = \frac{r\partial S_2(r)/\partial r}{(-S_3(r))^{2/3}} \quad (3)$$

independent of r in the whole region $0 < \varepsilon < 2$ which allows to find the physical values of \mathcal{S} and C through the simple relations

$$\mathcal{S} = - \left[\frac{2}{3Q(2)} \right]^{3/2}, \quad C = \frac{3}{2} \left[\frac{12}{d(d+2)} \right]^{2/3} Q(2). \quad (4)$$

The use of $S_3(r)$ in constructing universal quantities is advantageous because it can be found exactly from the spectral energy balance for all $\varepsilon < 2$:

$$S_3(r) = - \frac{3(d-1)\Gamma(2-\varepsilon)(r/2)^{2\varepsilon-3}D_0}{(4\pi)^{d/2}\Gamma(d/2+\varepsilon)}.$$

This expression allows, on one hand, to avoid calculation of graphs in construction of the ε expansion for $S_3(r)$, and, on the other, confirms that passing to the physical limit $\varepsilon \rightarrow 2$, in which $\Gamma(2-\varepsilon) \sim 1/(2-\varepsilon)$, requires the dependence $D_0 \simeq a(2-\varepsilon)$ to arrive at a finite value of $S_3(r)$. The choice of the coefficient a consistent with (2) yields the "4/5 law" of Kolmogorov: $S_3(r) = -\frac{4}{5}\mathcal{E}r$ [4, 15].

In the usual ε expansion at $d > 2$ the universal quantity $Q(\varepsilon)$ (3) has the form [16]

$$Q(\varepsilon) = \varepsilon^{1/3} \sum_{k=0}^{\infty} Q_k(d) \varepsilon^k. \quad (5)$$

The RG method allows to find successively the coefficients of $Q_k(d)$ as a result of calculation of renormalization constants and scaling functions in perturbation theory (loop expansion). In Refs. [5] - [14] only the one-loop approximation was used in the calculation of the Kolmogorov constant. The results of a two-loop calculation with the aid of the relations (4), (5) have been quoted in Ref. [16]. For the one-loop contribution to $Q_0(d)$ in Eq. (5) an analytic expression for all d may be obtained:

$$Q_0(d) = (1/3)[4(d+2)]^{1/3}. \quad (6)$$

The two-loop contribution $Q_1(d)$ gives rise to integrals which may be evaluated numerically for any particular values d . For $d = 3$ in the calculation of the Kolmogorov constant according to Eq. (4) the values $C^{(1)} = 1.47$ (one-loop approximation) and $C^{(2)} = 3.02$ (two-loop approximation) were obtained. Although the two-loop correction is not small, the recommended experimental value of the Kolmogorov constant $C \approx 2.0$ [4, 17] turned out to be in between the values given by the two approximations. Hardly any more could be expected in view of the fact that the value of the expansion parameter is not small. In what follows, we will show that the agreement with the experiment may be significantly improved by an approximate account of the high-order terms of the expansion (5).

Analysis of the dependence of the functions $Q_k(d)$ on the space dimension d shows that they have singularities at $d \leq 2$. In particular, $Q_k(d) \sim \Delta^{-k}$ for $2\Delta \equiv d-2 \rightarrow 0$. This means that in the course of d tending to 2 the expansion (5) necessarily will become "spoiled", because the relative contribution of the high-order terms will grow without limit. In the present two-loop approximation this feature shows in that the ratio $Q_1(d)/Q_0(d)$ in the limit $d \rightarrow \infty$ (far away from all singularities) is about 1/20 and monotonically grows with decreasing d assuming at $d = 3$ the value $\simeq 1/2$ of which the major part is brought about by graphs singular in the limit $2\Delta = d-2 \rightarrow 0$. This gives rise to hope that summation of leading Δ singularities in Eq. (5) allows to improve quantitative results of the RG theory.

In the theory of turbulence the space dimension $d = 2$ is exceptional from both the physical point of view (additional conservation laws, inverse energy cascade) and the formal procedure of UV renormalization, because in the limit $d \rightarrow 2$ new divergences appear in the graphs of the perturbation theory. These divergences show in the form of poles in Δ in the coefficients $Q_n(d)$ for $n \geq 1$ in Eq. (5). A consistent procedure to remove these divergences by an additional renormalization has been developed and gives rise to a two-parameter ε, Δ expansion [18]. In this Letter it is not possible to dwell on details of calculations. However, we want to point out the following principal issue.

The use of the ε, Δ expansion in the theory of turbulence was proposed in Ref. [19], whose author points out that an additional renormalization of the random force is required. This was carried out, following Ref. [1], by multiplicative renormalization of the random force. There is, however, a major difference between the models of Refs. [1] and [19]. In Ref. [1] a local correlation function of the random force is considered ($\sim k^2$ – model A, or $\sim \text{const}$ – model B), whereas in Ref. [19] the correlation function is nonlocal $\sim k^{4-d-2\varepsilon}$. From general theory of renormalization it is known that counter terms may only be local (see, e.g. [20, 21]), which means that the renormalization adopted in Ref. [19] is inconsistent. A consistent procedure, which we have used, was put forward in Ref. [18]. Our two-loop calculation allowed to confirm directly the general statement and show that it is not possible to renormalize the present model by multiplicative renormalization of the random force. We emphasize that, although this fact begins to show in the two-loop approximation, it renders the corresponding one-loop result incorrect as well, since the use of the RG approach is based on the existence of certain relations between all terms of the perturbation expansion which break down in an incorrect renormalization. A detailed discussion of this issue we defer to a separate publication.

In Refs. [18, 19] this two-parameter renormalization procedure (in one-loop approximation) was considered an alternative to the usual ε expansion. We exploit it in a different manner – as a way to improve the expansion (5) by carrying out an approximate summation of the high-order contributions.

To single out the leading poles, we express the coefficients $Q_k(d)$ in the form

$$Q_k(d) = \Delta^{-k} q_k(\Delta), \quad 2\Delta \equiv d - 2, \quad (7)$$

with a regular function

$$q_k(\Delta) = \sum_{l=0}^{\infty} q_{kl} \Delta^l. \quad (8)$$

Substitution of the expressions from Eqs.(7) and (8) in

Eq. (5) leads for the quantity Q to the representation

$$Q(\varepsilon) = \varepsilon^{1/3} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (\varepsilon/\Delta)^k q_{kl} \Delta^l. \quad (9)$$

The ε, Δ expansion corresponds to the asymptotic regime $\varepsilon \sim \Delta \rightarrow 0$, $\Delta/\varepsilon = \text{const}$. Hence, the quantities $(\varepsilon/\Delta)^k$ in Eq. (9) are not considered small and the powers Δ^l play the rôle of a formal small parameter. The quantity Q from Eq. (9) in the n th-order approximation ($n \geq 1$) is the series

$$\varepsilon^{1/3} \sum_{k=0}^{\infty} \sum_{l=0}^{n-1} (\varepsilon/\Delta)^k q_{kl} \Delta^l \equiv Q_{\varepsilon, \Delta}^{(n)}, \quad (10)$$

which corresponds to an approximate calculation of the coefficients (7) of the ε expansion (5) with the account of n terms in the sum (8). For a RG calculation of the quantity $Q_{\varepsilon, \Delta}^{(n)}$ in the ε, Δ -expansion scheme [18] an n -loop approximation would be needed.

Let us assume for the moment that we have carried out an n -loop calculation in the usual ε expansion thus determining the following partial sum of the series (5)

$$\varepsilon^{1/3} \sum_{k=0}^{n-1} Q_k(d) \varepsilon^k \equiv Q_{\varepsilon}^{(n)}, \quad (11)$$

and an n -loop calculation in the ε, Δ -expansion scheme as well, hence having determined the quantity $Q_{\varepsilon, \Delta}^{(n)}$ of Eq. (10). Then we have the possibility to amend the sum (11) by an approximate contribution of all higher powers of ε^k not taken into account in Eq. (11). The required information of this contribution is contained in the quantity $Q_{\varepsilon, \Delta}^{(n)}$. To obtain the improved value of Q we add the expressions (10) and (11), then subtract once the sum

$$\delta Q^{(n)} \equiv \varepsilon^{1/3} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} (\varepsilon/\Delta)^k q_{kl} \Delta^l$$

which enters twice in the sum of Eqs. (10) and (11). Thus, we arrive at the following n -loop approximation

$$Q_{eff}^{(n)} = Q_{\varepsilon}^{(n)} + Q_{\varepsilon, \Delta}^{(n)} - \delta Q^{(n)} \quad (12)$$

for Q . Our two-loop calculation yields the result

$$Q_{\varepsilon, \Delta}^{(1)} = 2 \left[\frac{2(\varepsilon + \Delta)^2 \varepsilon}{3(2\varepsilon + 3\Delta)^2} \right]^{1/3},$$

$$\frac{Q_{\varepsilon, \Delta}^{(2)}}{Q_{\varepsilon, \Delta}^{(1)}} = \left[1 + \left(0.5181\varepsilon + \frac{1}{6}\Delta \right) \right] \quad (13)$$

for the quantities $Q_{\varepsilon, \Delta}^{(1)}, Q_{\varepsilon, \Delta}^{(2)}$ with the subsequent expressions for $\delta Q^{(1)}, \delta Q^{(2)}$:

$$\delta Q^{(1)} = \frac{2}{3} (2\varepsilon)^{1/3},$$

$$\frac{\delta Q^{(2)}}{\delta Q^{(1)}} = \left(1 + \frac{2\varepsilon}{9\Delta} \right) \left[1 + \left(0.5181\varepsilon + \frac{1}{6}\Delta \right) \right]. \quad (14)$$

TABLE I: One and two-loop values of the Kolmogorov constant in the usual ε expansion (C_ε) and the double ε, Δ expansion ($C_{\varepsilon, \Delta}$); the contribution C_δ in Eq. (4) from the correction $\delta Q^{(n)}$ in Eq. (12), and the value C_{eff} from Eqs. (4), (12).

n	C_ε	$C_{\varepsilon, \Delta}$	C_δ	C_{eff}
1	1.47	1.68	1.37	1.79
2	3.02	3.57	4.22	2.37

Calculating at $d = 3$ the quantity $Q_\varepsilon^{(n)}$ from (11) with the aid of (6) and the value $Q_2(3) \simeq 0.4748$ found in [16], and substituting the result together with Eqs. (13) and (14) in Eq. (12) we find the quantity Q_{eff} in first and second approximation: $Q_{eff}^{(1)} = 1.38$, $Q_{eff}^{(2)} = 1.84$. Substitution of these values in Eq. (4) at $d = 3$ yields for the Kolmogorov constant and the skewness factor the values $C_{eff}^{(1)} = 1.79$, $C_{eff}^{(2)} = 2.37$, $S_{eff}^{(1)} = -0.33$, $S_{eff}^{(2)} = -0.22$.

In the Table I we have quoted for comparison the values of the Kolmogorov constant calculated according to Eq. (4) in the first and second order of the usual ε expansion (C_ε), the double ε, Δ expansion ($C_{\varepsilon, \Delta}$), the contribution C_δ in Eq. (4) from the correction $\delta Q^{(n)}$ in Eq. (12) and the value C_{eff} obtained from Eqs. (4) and (12). In all the cases quoted the recommended experimental value of the Kolmogorov constant $C \approx 2.0$ lies between the values of the first and second approximation. However, the difference between successive approximations is rather significant both in the ε expansion and in the ε, Δ expansion, let alone the leading terms of the ε expansion of the latter. For the improved ε expansion, i.e. for the quantity $C_{eff} = C_\varepsilon + C_{\varepsilon, \Delta} - C_\delta$ calculated according to Eqs. (12) and (4), however, this difference is about three times smaller leading to a far better agreement with the experimental data.

In conclusion, we have shown that a proper account of the "nearest singularity" in the coefficients of the ε expansion (5) leads to a significant improvement of the results of the two-loop RG calculation at $d = 3$. We have analysed the effect of this procedure at other d as well. It turned out to reduce significantly the relative contribution of the two-loop correction in the whole range considered $\infty > d \geq 2.5$. At the same time this contribution remained large at $d = 2$, which we think to be an effect of singularities at the next exceptional dimension $d = 1$.

Obviously, the proposed procedure of approximate summation of the ε expansion is applicable not only to the calculation of $Q(\varepsilon)$, but all universal quantities such as dimensions of composite operators.

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